IP sets and games

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Let’s pool our notes.

# Classic definitions

## Classical notions

We work in the partial semigroup .

The partial operation is disjoint union:

That is, when we write this means .

In other words: when we write or we always assume the product is defined (and any conditional statement where this occurs is restricted to the defined case).

Since we are interested in finite products, all sequences in are considered to be pairwise disjoint.

Note that for historic reasons we will sometimes consider the partial operation of ordered unions, i.e., we restrict the union operation not just to , but require .

#### Definition -set

is an -set if there exists (pairwise disjoint) such that .

Note: -sets are those that contain infinite -sets. (the “length” of an -set always describes the length of the sequence of generators).

The notion of -set that we will study has been designed as a transfinite approximation of the notion of -set.

## Defining -sets

There are three definitions for -sets. We will show later that they are equivalent. These are all from Beiglboeck-Towsner

#### The original inductive definition

We can define -sets inductively for (important: we’ll see that this hierarchy stops at , i.e., if is it is for all ).

* Every subset of is ,
* the set is iff is
* in case of limit ordinal the set is iff it is for all

Some easy things to check:

* Every non-empty set is .
* a set is () if and only if there exists a sequence such that Exercise! Check: where is inductively defined by , .
* In particular, an set can be characterized by having arbitrarily “long” finite FU-sets.

#### Definition using the Game

Define the game for .

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| I |  |  |  |  | ... |
| II |  |  |  |  | ... |

I.e., II plays decreasing sequences of ordinals and I plays sequences in . The game ends when II cannot continue.

Player I wins if is a sequence of pairwise disjoint sets and .

We will show that a set is iff I has a winning strategy in the game .

#### Definition using Trees

In the Beiglboeck-Towsner paper they introduce a notion of tree which is not a tree in the usual sense. This has to do with the messiness of addition on natural numbers. Since we circumvent this mess, we can actually consider trees in the classical (set theoretic) sense.

* A tree in our setting is always a subtree of , the disjoint, finite sequences in (with or without the root – it’s not relevant because we do not consider empty unions).
* The partial order on our trees is , but to avoid confusion we denote it by .
* If is well founded in the reverse order (i.e., does not have an infinite branch), we can define the usual (countable) rank on (short: leafs have rank and everything else the sup of (rank+1) of their successors).
* We will be interested in . For a colouring of , we say that is monochromatic if is.
* Given any set , we define the (possibly empty) tree .

We will show that a set is iff has rank .

# Basic Observations

## The equivalence of the three definitions

#### Theorem

TFAE

1. is
2. Player I has a winning strategy in
3. has rank

#### Proof

* 1) 2)
  + By induction ( is sort of meaningless for the game, so we skip that)
  + .
    - is iff by the exercise
    - II has only one move in , namely and the game ends.
    - So I has a winning strategy iff is non-empty – as desired.
    - is iff is
    - II has an optimal first move in , namely .
    - If is , we can describe a strategy for player .
      * I plays if II plays .
      * Otherwise, I plays according to the strategy we have (by induction hypothesis) for .
      * Check that this works.
    - If I has a winning strategy, then we can find with being .
      * Let be reaction to II playing .
      * Check that the strategy implies that is .
    - For the equivalence follows easily by induction hypothesis.
* 2) 3)
  + If player I has a winning strategy, then the results of the game according to the strategy form a tree of height .
  + This tree is a subtree of , hence has at least rank .
* 3) 2)
  + If has at least rank , we can describe a strategy for player I as follows.
  + II starts with some .
  + Since has at least rank , Player I can find an element of rank .
  + But is a sequence in ; pick any (in fact, any will do; but of course, might have length 1).
  + Inductively at move , player I can respond to by picking a successor of with
  + Remember that by induction hypothesis, .
  + Then I plays an element from .
  + Clearly, I will win the game since even .

## The hierarchy stops at .

In their paper, Mathias and Henry prove that is really an approximation of -set.

#### Theorem (Beiglboeck-Towsner)

is an -set iff is .

#### Proof

* The forward direction is easy – play an infinite sequence, you’ll always win the game.
* The reverse follows from: if is , then there exists such that is . With this in hand, you can simply follow the usual proofs of Hindman’s Theorem.
  + Otherwise, every is not so.
  + Hence there is a countable such that is not .
  + But is countable, is a regular ordinal, hence .
  + But then is not – a contradiction.

#### Corollary

In particular, is , then is for all .

#### Proof

* In the proof we have seen is iff is
* In other words, is iff is
* The claim follows by an easy induction.

## Some more stuff

Just to jot them down somewhere, we should mention:

We will almost always assume a slightly different meaning for set. Namely, in most proofs, is to imply ’not -set’.

# Canonical form

For certain , we can describe a canonical form of -sets.

Throughout this section means ’not ’.

* -sets (for some ) are essentially just for of lenght . In particular, the tree is just one branch of length .
  + It was an exercise in the Basics to show that any -set contains such an FP-set.
  + Clearly, such an FP-set is (the sequence gives a perfectly good strategy for -many moves)
* sets are essentially just a disjoint collection of FP-sets of length for all . In particular, the tree consists of countably many “disjoint” branches of increasing length.
  + By definition to be means to include arbitrarily large -sets.
  + By the above, these are just FP-sets of length .
  + Inductively, pick FP-sets which come in an ordered fashion, i.e, the generators .
    - Pick some .
    - Inductively having picked as ordered as desired, pick by first picking an FP-set of length .
    - Since the generators of that FP-set are pairwise disjoint, the last must have a minimum above – take those elements to form .
    - Check that this works.
* The canonical form of -sets (for )
  + Let be . We thin out inductively.
  + Consider .
  + This is an -set, so thin it out to its canonical form of “disjoint -sets”
  + In other words, wlog each has a unique maximal sequence such that
  + Now replace with .
  + These sets are precisely .
  + Continue this process!
  + In other words, can be pruned to look as follows:
    - First there’s a tree as for sets – longer and longer branches.
    - Each of its leafs then has a tree as for -sets.
      * And so forth, a total of n-times.

We will see later: -sets are the pairwise disjoint union of -sets.

And of course: -sets have the canonical form: for an infinite .

# is partition regular

## The Folkman-Rado-Sanders Theorem

Recall this classical result.

#### Theorem (Folkman, Rado and Sanders independently)

For every and any number of colors there exists the Folkman number such that: If has length then for every coloring , there exists homogeneous of length (i.e., restricted to is constant). In our terminology: whenever we -color an -set, we find a homogeneous set.

In their paper, Beiglboeck and Towsner give an extension of this theorem to -sets. We’ll discuss this later.

## sets are partition regular

An immediate consequence of the Folkman-Rado-Sanders Theorem can be phrased in our setting as follows.

#### Corollary

sets are partition regular.

#### Proof

* Let be , i.e., for all .
* Let and -coloring of .
* By Folkman-Rado-Sanders, we find homogeneous -sets for arbitrarily large .
* By pigeon hole principle, one color must appear infinitely often among these monochromatic -sets.
* In other words, is an -set.

We also know that -sets are partition regular: after all, we have shown that -sets are exactly IP-sets, so HIndman’s Theorem gives us partition regularity.

The question is whether any other type of -set is partition regular. We will see later that there are club many such .

## -sets are partition regular

The simple, but key idea will be that any -coloring of an will have a monochromatic subset. Then all we have to do is apply the pigeon hole principle just as for partition regularity of -sets. We give a different proof than Beiglboeck-Towsner for this that could generalize better to larger ordinals.

#### Block-homogeneous strategies

To be able to speak about things efficiently, we need an auxiliary notion, the concept of a block-homogeneous strategy.

#### Definition

Consider , an -set, a coloring .  
Consider a play .  
Then for some .  
We say that player I has a block-homogeneous strategy if for all plays according to the strategy

whenever   
In other words, the color of only depends on the combination of -blocks of II’s moves that come up in the indices of the element’s generators.  
Note that a block-homegeneous strategy on an -set comes with a color scheme that encodes the colors of all possible combinations of -blocks.

It seems to be key to understand how to generalize this notion to higher ordinals.

The key for our argument will be the following lemma.

#### Lemma

If is , then for every and every -coloring, player I has a block-homogeneous strategy for .

This is somewhat surprising since one would expect it to be harder to play a block-homogeneous match. I’m guessing a generalization will forces us to reduce the rank a little.

#### Proof

* We essentially use the canonical form of -sets.
* Consider
* By assumption on , we know that is an -set.
* Thin it out to become a disjoint union of -sets generated by sequences .
* Remember each is in exactly one .
* Also, for each we know that is .
* Now for each define a coloring by coding the colors of each together with every (for ).
* Note that even though we have increased the number of colors, we can apply our induction hypothesis since it works for all numbers of colors.
* By induction hypothesis, we have a block-homogeneous strategy for each .
* Recall that this means we have, for each a scheme that codes how the block-homegeneous strategies guarantee their colors.
* But this means we have a coloring mapping each point to its (coded) scheme.
* Since -sets are partition regular, we find a homogeneous which is . Note that this also means that is homogeneous (which we could also ensure by refining it once more via the original ).
* Thin out to its canonical form.
* We can now describe a strategy for player I.
  + If II begins with choose the unique -long FP-set in .
  + While II plays above , player I plays from that sequence.
  + Note that this sequence is a subset of one of the from our original (thanks to our canonical form).
  + When II plays below , play according to the strategy from .
* Since our moves above were in , we know that they agree on that strategy.
* In other words, our whole strategy is block-homogeneous.

With this lemma, we can calculate some transfinite Folkman-numbers.

#### Lemma

Whenever is and , then we find a homogeneous subset of .  
In other words, .

#### Proof

* By the previous lemma, we have a block homogeneous strategy for
* This means the associated color scheme is a map .
* Since is the Folkman number, we can find such that is -monochromatic.
  + The elements and their unions should be thought of as a “hint” while playing an -game: at the end look at combinations where something from each -block noted by appears.
  + Note that the -homogeneity means that the block-homogeneous strategy is monochrome for combinations that follow this “hint”.
* We can now describe a strategy for player I using the block-homogeneous strategy and the “hints” from .
  + II plays some .
  + Player I starts an game on the side as follows:
    - Pretend II plays .
    - The resulting play yields a sequence of length .
    - Turn these into -many sets, unioning up every th element.
  + Now back to the actual game
  + While II plays above , player I responds with the -many elements we just build on the side.
  + When II drops below , player I continue using the remaining in the same manner (but possibly with other values for ).
* By the use of a block-homegeneous strategy and the choice of the , player I will play a homogeneous sequence.

Before we proceed to prove the theorem, let us note a small corollary.

#### Corollary

The canonical form of an -set can be sketched as “pairwise disjoint union of (canonical) -sets”

#### Proof

* Inductively choose such sets.
* At each step pick an set so large, that removing the previously chosen sets leaves us with an -set (this is possible because it corresponds to a partition into parts)

With the lemma we can easily derive our theorem.

#### Theorem

-sets are partition regular.

#### Proof

* Let be and .
* By the above lemma, we find arbitrarily large monochromatic -subsets.
* By the pigeon hole principle, we finde one color with arbitrarily large monochromatic -sets.
* Hence is an -set.

# Visual buffer between Peter’s and Francois’s notes

Let be such that is infinite for each . Let be the family of all nonempty finite subsets of such that and .

If is a limit ordinal then the unordered -rank of is .

For , let be the maximal size of a set such that

* ,
* ,
* .

Then define the -rank of to be the ordinal .

If are disjoint and then .

Write and , where . Without loss of generality, we may assume that and hence that .

*Case .* Then satisfies the three conditions in the definition of . It follows at once that and hence .

*Case .* Again, satisfies the three conditions in the definition of . It follows that and hence . To see that , consider the set

This is a possibly empty set satisfies the three conditions in the definition of . It follows that . Now observe that

It follows that

$${\operatorname{rank}\nolimits}\_f(x \cup y) \leq \omega\xi + p + n(y) + (q - p - 1) = \omega\zeta + (q + n(y) - 1) < {\operatorname{rank}\nolimits}\_f(y).\qedhere$$

For every there is a such that .

Write where . Let and let witnesses this fact. We consider three cases.

*Case and .* Let . Note that and . Hence .

*Case and .* Let where and for . Note that and . Hence .

*Case .* First pick such that . Then let be a set of maximal size such that and . Note that and hence .

Now consider a play in the unordered game .

Note that the sets are such that

If , then can play and to win. If , then can play so that to win.

Let be a limit ordinal. Define and by the equation . Let be such that is infinite for each and let as above. Define the partition by

and

The -rank of is .

For , let be the maximal size of a set such that

* ,
* ,
* .

Define the rank function .

If are disjoint and then .

Similar to Lemma ???.

For every there is a such that .

Similar to Lemma ???.

Now consider a play in the unordered game .

Note that the sets are such that

If , then can play and to win. If , then can play so that to win.

The -rank of is no more than .

For , let be the maximal size of a set such that

* ,
* ,
* .

Note that the first and last clauses entail that , so there are only finitely many sets which satisfy the three conditions. Define the rank function .

If are disjoint and then .

Without loss of generality, we may assume that .

Note that we then have which means that . So we need to show that .

Since , we know that

is a nonempty subset of which satisfies the three conditions in the definition of . It follows at once that , as required.

Now consider a play in the unordered game .

Note that the sets are such that

If , then can play and to win.

Let us say that is *closed under subunions* if is a disjoint sequence of elements of such that , then the union of any subsequence of is in too.

Suppose is closed under subunions. If player has a winning strategy in the game with first move , then admits a rank function into .

Let be player II’s winning strategy. Given a set , let be the minimum value where ranges over all the disjoint partitions of with elements of . We clearly have for all .

To see that is a rank function, suppose that and are disjoint elements of . Let be a disjoint partition of with elements of such that . Note that since is closed under subunions, is a valid sequence of moves for player in . Then is a disjoint partition of with elements of . Again, since is closed under subunions, is a valid sequence of movers for player in . Therefore,

A symmetric argument shows that .

Suppose that is a tree of height at least and that . If is a coloring of such that for every there is a such that , then one of the following holds:

1. There is a tree with height at least such that for every there is a such that .
2. There is a and a tree with height at least such that for every there is a such that for every .

By indution on . The case is trivial and so are limit cases.

Suppose that has height at least . Let $T\_0 = \{ t \in T : \height(t) \geq \alpha\cdot\beta\}$ then is a tree of height at least . If satisfy condition , we’re done. Otherwise, there is an such that for every there is some such that .

Apply the induction hypothesis to the tree , , , which is a tree of height at least .

On the one hand, if there is a tree of height at least such that for every there is a such that . Then has height at least and it satisfies  since .

On the other hand, if there are a and a tree of height at least for every there is a such that for every . Then and are as required

Suppose that is a tree of height at least and that . If is a coloring of , then one of the following holds:

1. There is a tree with height at least such that for every there is a such that .
2. There is a and a tree with height at least such that for every there is a such that for every .

Suppose that is a tree of height greater than and that is a -coloring of . Then there are a and a tree with height at least such that for every there is a such that .

We apply the previous corollary repeatedly. First, pick such that has height at least . Apply the corollary to , to obtain either the required tree of height , or a tree of height together with a wuch that for every there is a such that for every . In the latter case, we apply the corollary again to . And so on until a suitable tree of height is found.

If we never find the required tree of height , then we end up with a tree of height such that for every and every there is a such that for all . In particular, we obtain a sequence such that

whenever .